

Dark-like states for the multi-qubit and multi-photon Rabi models

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Abstract. There are well-known dark states in the even-qubit Dicke models, which are the products of the two-qubit singlets and a Fock state, where the qubits are decoupled from the photon field. These spin singlets can be used to store quantum correlations since they preserve entanglement even under dissipation, driving and dipole-dipole interactions. One of the features for these dark states is that their eigenenergies are independent of the qubit-photon coupling strength. We have obtained a novel kind of dark-like states for the multi-qubit and multi-photon Rabi models, whose eigenenergies are also constant in the whole coupling regime. Unlike the dark states, the qubits and photon field are coupled in the dark-like states. Furthermore, the photon numbers are bounded from above commonly at 1, which is different from that for the one-qubit case. The existence conditions of the dark-like states are simpler than exact isolated solutions, and may be fine tuned in experiments. While the single-qubit and multi-photon Rabi model is well-defined only if the photon number $M \leq 2$ and the coupling strength is below a certain critical value, the dark-like eigenstates for multi-qubit and multi-photon Rabi model still exist, regardless of these constraints. In view of these properties of the dark-like states, they may find similar applications like “dark states” in quantum information.

1. Introduction

The Rabi model [1] has been born for 80 years [2]. With semi-classical [1] and fully quantized versions [3], it has found wide applications in magnetic resonance [1], solid state [4], quantum optics [5], cavity QED [6], circuit QED [7] and quantum information [8]. Although the quantum Rabi model has a very simple form, describing the simplest interaction between light and matter, its analytical solution had not been found until 2011 by Braak [9]. This is partly due to the fact that there is no closed subspace in its Fock space, which is different from that in the Jaynes-Cummings model [3] with the rotating wave approximation [10]. The qubit-photon ultrastrong coupling regime has been reached in recent circuit QED experiment [11]. However, in this regime, the Jaynes-Cummings model fails, so many researches then focus on the Rabi model, which include the analytical solution of the Rabi model retrieved by Chen et al using Bogoliubov operators [12], two-photon [13–15], two-qubit [16–20] and multi-qubit [21–23] generalizations, exact real time dynamics [24, 25], deep strong coupling [26], anisotropic Rabi model [27] and so on [28–30].

For the single qubit Rabi model, the eigenstates consist of infinite photon number states, because there are no closed subspaces in the Fock space [9]. But this is not the case for the multi-qubit case, because more qubits will bring closed subspaces. For example, Rodríguez-Lara et al has found “trapping states” (“dark states”) [31] in the even qubit Dicke model, where two identical qubits form a spin singlet and the eigenstates are just products of these singlets and a Fock state. These singlets are decoupled from the photon field, and will survive even under dissipation, driving and also dipole-dipole interactions. So they can be used to store quantum correlations. Since the qubits and photon are decoupled, the eigenenergies of the “dark states” are constants in the whole qubit-photon coupling regime, which correspond to horizontal lines in the spectrum.

There has been some researches on the “dark-like” states of the two-qubit and single photon Rabi model [18, 32]. In this paper, we will study multi-qubit and multi-photon Rabi model, and show that “dark-like” eigenstates commonly exist, surprisingly not just for even-qubit, but also odd-qubit, and multi-photon cases. The single-qubit and multi-photon Rabi model is well-defined only if the photon number $M \leq 2$ and the coupling strength is below a certain critical value [33], but with multi-qubit it will bring about closed subspaces and the dark-like eigenstates still exist in the whole coupling regime and for $M > 2$. These dark-like states possess several features. Firstly, they exist in the whole coupling regime with constant eigenenergies, just like the “dark states”. But surprisingly, the qubit and photon are not decoupled and the wavefunctions are coupling dependent. Secondly, the photon numbers in the eigenstates are bounded from above at K . In particular, $K = 1$ for the single-photon case. Thirdly, their existence conditions are simpler than exact isolated solutions, because they can be realized in arbitrary coupling regime with the same qubit energy, which may be fine tuned in experiment. So just like the “dark states”, these “dark-like” states may get possible application in quantum information.

The paper is organized as follows. In section 2, we search for the “dark-like” eigenstates for the multi-qubit Rabi model. In section 3, we generalize our study to the multi-qubit

and multi-photon Rabi models. In section 4 we give some experimental considerations for the implementation in quantum controllable platforms. Finally, we give our conclusions in section 5.

2. Dark-like states for the multi-qubit Rabi model

The Hamiltonian of the N-qubit quantum Rabi model reads ($\hbar = 1$) [17, 18]

$$H_{NQ} = \omega a^\dagger a + \sum_{i=1}^N g_i \sigma_{ix} (a + a^\dagger) + \sum_{i=1}^N \Delta_i \sigma_{iz}, \quad (1)$$

where a^\dagger and a are the single mode photon creation and annihilation operators with frequency ω , respectively. $\sigma_{i\alpha}$, ($\alpha = x, y, z$) are the Pauli matrices corresponding to the i -th qubit. $2\Delta_i$ is the energy level splitting of the i -th qubit, and g_i is the qubit-photon coupling constant. ω is set to 1 in the following discussion.

The Hamiltonian (1) is usually infinite dimensional in the Fock space, which is exactly the case for just one qubit, but with more qubits it will bring about possible closed subspace. Working on this finite dimensional subspace, we can obtain the solution of the Hamiltonian (1) with finite photon numbers and the dark-like eigenstates. For this purpose, we must first search for the existence condition of this closed subspace. H_{NQ} possesses a \mathbb{Z}_2 symmetry with the transformation $R = \exp(i\pi a^\dagger a) \prod_{i=1}^N \sigma_{iz}$, so we have

$$R|p\rangle = p|p\rangle \quad (2)$$

with $p = \pm 1$. At the same time, we can categorize the N-qubit states $\{|\psi\rangle_{Nq}\}$ into two sets with the eigenvales of $\prod_{i=1}^N \sigma_{iz}$ being 1 and -1 respectively, and they are denoted by 2^{N-1} dimensional row vectors $(|\psi\rangle_{Nq+})$ and $(|\psi\rangle_{Nq-})$. It is easy to find the following relations

$$(|\psi\rangle_{Nq+}) = (|\psi\rangle_{N-1 q-} \otimes |\downarrow\rangle_N, |\psi\rangle_{N-1 q+} \otimes |\uparrow\rangle_N), \quad (3)$$

$$(|\psi\rangle_{Nq-}) = (|\psi\rangle_{N-1 q+} \otimes |\downarrow\rangle_N, |\psi\rangle_{N-1 q-} \otimes |\uparrow\rangle_N), \quad (4)$$

with the initial states $|\psi\rangle_{1q+} = |\uparrow\rangle_1$ and $|\psi\rangle_{1q-} = |\downarrow\rangle_1$. Then we have two unconnected subspaces

$$|0, \psi_{Nq+}\rangle \leftrightarrow |1, \psi_{Nq+}\rangle \leftrightarrow |2, \psi_{Nq+}\rangle \leftrightarrow \dots \quad (p = 1) \quad (5)$$

$$|0, \psi_{Nq-}\rangle \leftrightarrow |1, \psi_{Nq-}\rangle \leftrightarrow |2, \psi_{Nq-}\rangle \leftrightarrow \dots \quad (p = -1) \quad (6)$$

Only neighboring states within each parity chain are connected, so H_{NQ}^\pm will take the following form in even ($p = +1$) or odd ($p = -1$) subspace

$$H_{NQ}^\pm = \begin{pmatrix} D_{N0}^\pm & O_{N0}^\pm & 0 & 0 & 0 & \dots \\ O_{N0}^\pm & D_{N1}^\pm & O_{N1}^\pm & 0 & 0 & \dots \\ 0 & O_{N1}^\pm & D_{N2}^\pm & O_{N2}^\pm & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (7)$$

where D_{Nj}^\pm and O_{Nj}^\pm ($j = 0, 1, 2, 3, \dots$) can be written as

$$D_{Nj}^\pm = (\langle j, \psi_{Nq\pm(-1)^j} |)^T H_{NQ} (|j, \psi_{Nq\pm(-1)^j} \rangle), \quad (8)$$

$$O_{Nj}^\pm = (\langle j+1, \psi_{Nq\mp(-1)^j} |)^T H_{NQ} (|j, \psi_{Nq\pm(-1)^j} \rangle), \quad (9)$$

where $(|j, \psi_{Nq\pm(-1)^j}\rangle)$ is a 2^{N-1} dimensional vector, and D_{Nj}^\pm, O_{Nj}^\pm are $2^{N-1} \times 2^{N-1}$ matrixes. Substituting Eqs. (3) and (4) into Eqs. (8) and (9), we get the following expressions for D_{Nj}^\pm and O_{Nj}^\pm ,

$$D_{Nj}^\pm = (\langle j, \psi_{N-1 q\mp(-1)^j} | \otimes_N \langle \downarrow |, \langle j, \psi_{N-1 q\pm(-1)^j} | \otimes_N \langle \uparrow | \rangle^T \times H_{NQ}(|j, \psi_{N-1 q\mp(-1)^j}\rangle \otimes | \downarrow \rangle_N, |j, \psi_{N-1 q\pm(-1)^j}\rangle \otimes | \uparrow \rangle_N), \quad (10)$$

$$O_{Nj}^\pm = (\langle j+1, \psi_{N-1 q\pm(-1)^j} | \otimes_N \langle \downarrow |, \langle j+1, \psi_{N-1 q\mp(-1)^j} | \otimes_N \langle \uparrow | \rangle^T \times H_{NQ}(|j, \psi_{N-1 q\mp(-1)^j}\rangle \otimes | \downarrow \rangle_N, |j, \psi_{N-1 q\pm(-1)^j}\rangle \otimes | \uparrow \rangle_N) \quad (11)$$

where

$$H_{NQ} = H_{N-1 Q} + \Delta_N \sigma_{Nz} + g_N \sigma_{Nx} (a + a^\dagger). \quad (12)$$

Then we have

$$D_{Nj}^\pm = \begin{pmatrix} D_{N-1 j}^\mp - \Delta_N & 0 \\ 0 & D_{N-1 j}^\pm + \Delta_N \end{pmatrix}, \quad (13)$$

$$O_{Nj}^\pm = O_{Nj}^\mp = O_{Nj} = \begin{pmatrix} O_{N-1 j} & \sqrt{j+1} g_N I \\ \sqrt{j+1} g_N I & O_{N-1 j} \end{pmatrix}, \quad (14)$$

with the initial condition

$$D_{1j}^\pm = j \pm (-1)^j \Delta_1, \quad (15)$$

$$O_{1j}^\pm = \sqrt{j+1} g_1. \quad (16)$$

As seen from Eq. (14), generally there is no closed subspace if O_{Nj} is nontrivial, which is exactly the case for single qubit with $g \neq 0$. But for the multi-qubit case, O_{Nj} can be equivalently trivial even for non-zero coupling constant g_i , if its eigenvalues are 0, which leads to the closed subspaces. Suppose that a subspace is spanned by $\{|J, \psi_{Nq\pm(-1)^J}\rangle, |J+1, \psi_{Nq\pm(-1)^{J+1}}\rangle, \dots, |K, \psi_{Nq\pm(-1)^K}\rangle\}$. If

$$O_{N J-1} c_{N,J}^\pm |J, \psi_{Nq\pm(-1)^J}\rangle = 0, \quad (17)$$

$$O_{N K} c_{N,K}^\pm |K, \psi_{Nq\pm(-1)^K}\rangle = 0, \quad (18)$$

where $c_{N,J}^\pm$ and $c_{N,K}^\pm$ are coefficients of $|J, \psi_{Nq\pm(-1)^J}\rangle$ and $|K, \psi_{Nq\pm(-1)^K}\rangle$ respectively, then this subspace is closed. Each of $c_{N,J}^\pm$ and $c_{N,K}^\pm$ contains 2^{N-1} components since $|J, \psi_{Nq\pm(-1)^J}\rangle$ and $|K, \psi_{Nq\pm(-1)^K}\rangle$ are 2^{N-1} dimensional vectors. So combined with the eigenvalue equation of H_{NQ} in this closed subspace, we obtain

$$\begin{pmatrix} O_{N J-1} & 0 & 0 & 0 & \dots \\ D_{N J-1}^\pm - E^\pm & O_{N J} & 0 & 0 & \dots \\ O_{N J} & D_{N J+1}^\pm - E^\pm & O_{N J+1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & O_{N K-2} & D_{N K-1}^\pm - E^\pm & O_{N K-1} \\ 0 & \dots & 0 & O_{N K-1} & D_{N K}^\pm - E^\pm \\ 0 & \dots & 0 & 0 & O_{N K} \end{pmatrix} \begin{pmatrix} c_{N,J}^\pm \\ c_{N,J+1}^\pm \\ \dots \\ c_{N,K-1}^\pm \\ c_{N,K}^\pm \end{pmatrix} = 0. \quad (19)$$

Clearly, there are more equations (rows) than variables (columns) in this system of linear homogeneous equations, so only for some special conditions, we may obtain a solution with finite photon numbers. We can use elementary row transformations to reduce the matrix

into row echelon form, then if the number of the non-zero rows is less than that of the columns, there will be non-trivial solutions. At the same time, Eqs. (17) and (18) are decoupled from other equations in Eq. (19), which is just the existence condition of the closed subspace, and they differ only by a constant. We can eliminant all the constants and define $O_N = O_{NJ}/\sqrt{J+1}$, then Eqs. (17) and (18) can be equivalent to the statement that the eigenvalues of O_n are zero, and both $c_{N,J}^\pm |J, \psi_{Nq\pm(-1)^J}\rangle$ and $c_{N,K}^\pm |K, \psi_{Nq\pm(-1)^J}\rangle$ are its null vectors.

O_N takes different forms for qubit number N , but we can find a unified form for its eigenvalues by analyzing its determinant

$$\begin{aligned} |O_N| &= \left| \begin{pmatrix} O_{N-1} & g_N \\ g_N & O_{N-1} \end{pmatrix} \right| = \left| \begin{pmatrix} O_{N-1} & g_N \\ g_N - O_{N-1} & O_{N-1} - g_N \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} O_{N-1} + g_N & 0 \\ g_N - O_{N-1} & O_{N-1} - g_N \end{pmatrix} \right| = |O_{N-1} + g_N| |O_{N-1} - g_N|. \quad (20) \end{aligned}$$

So if the eigenvalues of O_{N-1} are $e_{N-1,i}$ ($i = 1, 2, \dots, 2^{N-2}$), then the eigenvalues of O_N would be $e_{N-1,i} + g_N$ and $e_{N-1,i} - g_N$ with the initial condition $e_{1,1} = g_1$. Some eigenvalues and eigenvectors of O_N are shown in table 1.

Table 1. Qubit number N , eigenvalues and corresponding eigenvectors of O_N .

| N | eigenvalues | transpose of the eigenvectors |
|-----|-------------------------|--------------------------------|
| 2 | $g_1 + g_2$ | $(1, 1)$ |
| 2 | $g_1 - g_2$ | $(1, -1)$ |
| 3 | $g_1 - g_2 - g_3$ | $(1, -1, -1, 1)$ |
| 3 | $g_1 + g_2 - g_3$ | $(-1, -1, 1, 1)$ |
| 3 | $g_1 - g_2 + g_3$ | $(-1, 1, -1, 1)$ |
| 3 | $g_1 + g_2 + g_3$ | $(1, 1, 1, 1)$ |
| 4 | $g_1 - g_2 - g_3 - g_4$ | $(-1, 1, 1, -1, 1, -1, -1, 1)$ |
| 4 | $g_1 + g_2 - g_3 - g_4$ | $(1, 1, -1, -1, -1, -1, 1, 1)$ |
| 4 | $g_1 - g_2 + g_3 - g_4$ | $(1, -1, 1, -1, -1, 1, -1, 1)$ |
| 4 | $g_1 + g_2 + g_3 - g_4$ | $(-1, -1, -1, -1, 1, 1, 1, 1)$ |
| 4 | $g_1 - g_2 - g_3 + g_4$ | $(1, -1, -1, 1, 1, -1, -1, 1)$ |
| 4 | $g_1 + g_2 - g_3 + g_4$ | $(-1, -1, 1, 1, -1, -1, 1, 1)$ |
| 4 | $g_1 - g_2 + g_3 + g_4$ | $(-1, 1, -1, 1, -1, 1, -1, 1)$ |
| 4 | $g_1 + g_2 + g_3 + g_4$ | $(1, 1, 1, 1, 1, 1, 1, 1)$ |

Setting the eigenvalues of O_n to be 0, which just depends on the coupling strength, and $c_{N,J}^\pm |J, \psi_{Nq\pm(-1)^J}\rangle$, $c_{N,K}^\pm |K, \psi_{Nq\pm(-1)^J}\rangle$ to be its null vectors, we can simplify (19). Now the relations between all the components of each of $c_{N,J}^\pm$ and $c_{N,K}^\pm$ are fixed, so there is only 1 variable. Meanwhile, using $O_{N,J-1}c_{N,J}^\pm = 0$ and $O_{N,K}c_{N,K}^\pm = 0$ to simplify Eq. (19) by elementary row transformation and then put them aside, we obtain a necessary but not

sufficient condition for a solution

$$\begin{vmatrix} D_{N J}^{\pm} - E^{\pm} & O_{N J} & 0 & 0 & \dots \\ 0 & D_{N J+1}^{\pm} - E^{\pm} & O_{N J+1} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & O_{N K-2} & D_{N K-1}^{\pm} - E^{\pm} & 0 \\ 0 & \dots & 0 & O_{N K-1} & D_{N K}^{\pm} - E^{\pm} \end{vmatrix} = 0, \quad (20)$$

which is generally dependent both on the qubit energy and coupling strength, except for $K = J + 1$. But if $J \neq 0$, the corresponding wavefunction will not depend on the coupling strength at all and it turns into the “dark state”. So in order to search for the dark-like states, we just need to consider the case of $J = 0$ and $K = 1$. The equations which determine the solution to H_{NQ} reads

$$\begin{pmatrix} D_{N0}^{\pm} - E^{\pm} & 0 \\ O_{N0} & D_{N1}^{\pm} - E^{\pm} \\ 0 & O_{n1} \end{pmatrix} \begin{pmatrix} c_{N,0}^{\pm} \\ c_{N,1}^{\pm} \end{pmatrix} = 0. \quad (22)$$

Solving this equation is the key point to obtaining the dark-like eigenstates for the multi-qubit and multi-photon Rabi models.

Let us start with the simplest case of $N = 2$. For this case, we have $(|\psi\rangle_{2q+}) = (|\downarrow, \downarrow\rangle, |\uparrow, \uparrow\rangle)$, $(|\psi\rangle_{2q-}) = (|\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle)$, and

$$O_2 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}, \quad (23)$$

whose eigensystem is shown in table 1. Choosing $g_1 = g_2 = g/2$ and $c_{2,1,1} = -c_{2,1,2}$ to simplify (22), we arrive at

$$\begin{pmatrix} \mp\Delta_1 - \Delta_2 - E^{\pm} & 0 & 0 \\ 0 & \pm\Delta_1 + \Delta_2 - E^{\pm} & 0 \\ g/2 & g/2 & 1 \pm \Delta_1 - \Delta_2 - E^{\pm} \\ g/2 & g/2 & -1 \pm \Delta_1 - \Delta_2 + E^{\pm} \end{pmatrix} \begin{pmatrix} c_{2,0,1}^{\pm} \\ c_{2,0,2}^{\pm} \\ c_{2,1,1}^{\pm} \end{pmatrix} = 0. \quad (24)$$

After elementary row transformation, the coefficient matrix in Eq. (24) is simplified to the form

$$\begin{pmatrix} g/2 & g/2 & 1 \pm \Delta_1 - \Delta_2 - E^{\pm} \\ \mp\Delta_1 - \Delta_2 - E^{\pm} & 0 & 0 \\ 0 & \pm\Delta_1 + \Delta_2 - E^{\pm} & 0 \\ 0 & 0 & E^{\pm} - 1 \end{pmatrix} \quad (25)$$

There are three columns, so only two non-zero rows can exist in its row echelon form, from which we obtain

$$\Delta_1 + \Delta_2 = E^+ = 1, \quad (26)$$

with eigenstate

$$|\psi\rangle_e = \frac{1}{\mathcal{N}} \left(\frac{2(\Delta_1 - \Delta_2)}{g} |0, \uparrow, \uparrow\rangle - |1, \uparrow, \downarrow\rangle + |1, \downarrow, \uparrow\rangle \right), \quad (27)$$

for even parity and

$$\Delta_1 - \Delta_2 = E^- = 1, \text{ or } \Delta_2 - \Delta_1 = E^- = 1 \quad (28)$$

with eigenstates

$$|\psi\rangle_{g1} = \frac{1}{\mathcal{N}} \left(\frac{2(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right), \quad (29)$$

$$|\psi\rangle_{g2} = \frac{1}{\mathcal{N}} \left(\frac{2(\Delta_1 + \Delta_2)}{g} |0, \downarrow, \uparrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right), \quad (30)$$

respectively, for odd parity. Eigenstates (27), (29), (30) exist for any coupling strength $g_1 = g_2 = g/2$ with constant eigenenergy $E^\pm = 1$, corresponding to a horizontal line in the spectra, which has been shown in Ref. [18]. These properties are just like those for the “dark state” formed by the qubit singlet. However, for these eigenstates, the qubit and photon are not decoupled, and the photon number is bounded from above at 1.

Then we consider the case of 3 qubit, where $(|\psi\rangle_{3q+}) = (|\downarrow, \downarrow, \downarrow\rangle, |\uparrow, \uparrow, \downarrow\rangle, |\uparrow, \downarrow, \uparrow\rangle, |\downarrow, \uparrow, \uparrow\rangle)$, $(|\psi\rangle_{3q-}) = (|\uparrow, \downarrow, \downarrow\rangle, |\downarrow, \uparrow, \downarrow\rangle, |\downarrow, \downarrow, \uparrow\rangle, |\uparrow, \uparrow, \uparrow\rangle)$, and

$$O_3 = \begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ g_2 & g_1 & 0 & g_3 \\ g_3 & 0 & g_1 & g_2 \\ 0 & g_3 & g_2 & g_1 \end{pmatrix}, \quad (31)$$

whose eigensystem is shown in table 1. For $g_1 = g_2 + g_3$, $g_2 = g_1 + g_3$, or $g_3 = g_1 + g_2$, the eigenvalues are zero, and corresponding eigenvectors are nullvectors. $(D_{30}^\pm - E^\pm)c_{3,0}^\pm = 0$ in Eq. (22) is decoupled from other parts. The coefficient matrix $D_{30}^\pm - E^\pm$ takes the diagonal form

$$\begin{pmatrix} \pm\Delta_1 - \Delta_2 - \Delta_3 - E^\pm & 0 & 0 & 0 \\ 0 & \mp\Delta_1 + \Delta_2 - \Delta_3 - E^\pm & 0 & 0 \\ 0 & 0 & \mp\Delta_1 - \Delta_2 + \Delta_3 - E^\pm & 0 \\ 0 & 0 & 0 & \pm\Delta_1 + \Delta_1 + \Delta_3 - E^\pm \end{pmatrix}. \quad (32)$$

Choosing $g_1 = g_2 + g_3$ and $c_{3,1,1} = -c_{3,1,2} = -c_{3,1,3} = c_{3,1,4}$ to simplify the other part

$$\begin{pmatrix} O_{30} & D_{31}^\pm - E^\pm \\ 0 & O_{31} \end{pmatrix} \begin{pmatrix} c_{3,0}^\pm \\ c_{3,1}^\pm \end{pmatrix} = 0, \quad (33)$$

we arrive at

$$\begin{pmatrix} g_2 + g_3 & g_2 & g_3 & 0 & 1 \mp \Delta_1 - \Delta_2 - \Delta_3 - E^\pm \\ g_2 & g_2 + g_3 & 0 & g_3 & -(1 \pm \Delta_1 + \Delta_2 - \Delta_3 - E^\pm) \\ g_3 & 0 & g_2 + g_3 & 0 & -(1 \pm \Delta_1 - \Delta_2 + \Delta_3 - E^\pm) \\ 0 & g_3 & 0 & g_2 + g_3 & 1 \mp \Delta_1 + \Delta_1 + \Delta_3 - E^\pm \end{pmatrix} \begin{pmatrix} c_{3,0,1}^\pm \\ c_{3,0,2}^\pm \\ c_{3,0,3}^\pm \\ c_{3,0,4}^\pm \\ c_{3,1,1}^\pm \end{pmatrix} = 0. \quad (34)$$

After elementary row transformation, the coefficient matrix in Eq. (34) becomes

$$\begin{pmatrix} 1 & 0 & 0 & -1 & \frac{1-\Delta_3-E^\pm}{g_3} + \frac{1-\Delta_2-E^\pm}{g_2} \\ 0 & 1 & 0 & 1 & \frac{-1+\Delta_3+E^\pm}{g_3} + \frac{-1\mp\Delta_1+E^\pm}{g_2} \\ 0 & 0 & 1 & 1 & \frac{-1+\Delta_2+E^\pm}{g_2} + \frac{-1\mp\Delta_1+E^\pm}{g_2+g_3} \\ 0 & 0 & 0 & 0 & 4 - 4E^\pm \end{pmatrix} \quad (35)$$

There are totally 5 variables, so the total nonzero rows in Eqs. (32) and (35) should be less than 5. By choosing $E^\pm = 1$, the nonzero rows in Eq. (35) reduce to 3, which means only one nonzero row can exist in Eq. (32) to obtain a nontrivial solution. Luckily, there is one such case for odd parity with the following parameters

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta = E^- = 1. \quad (36)$$

Substituting Eq. (36) into Eqs. (32) and (35), we obtain a dark-like state

$$|\psi\rangle = \frac{g_3}{g_2(g_2 + g_3)}|0, \uparrow, \uparrow, \downarrow\rangle + \frac{g_2}{g_3(g_2 + g_3)}|0, \uparrow, \downarrow, \uparrow\rangle - \frac{g_2 + g_3}{g_2 g_3}|0, \downarrow, \uparrow, \uparrow\rangle \\ + |1, \uparrow, \downarrow, \downarrow\rangle - |1, \downarrow, \uparrow, \downarrow\rangle - |1, \downarrow, \downarrow, \uparrow\rangle + |1, \uparrow, \uparrow, \uparrow\rangle \quad (37)$$

If we choose $g_2 = g_1 + g_3$ and $-c_{3,1,1} = c_{3,1,2} = -c_{3,1,3} = c_{3,1,4}$ to satisfy the condition $O_{3,1}c_{3,1}^\pm = 0$, the corresponding solution can be obtained just by interchanging the states of the first and second qubits, including the coupling strength in Eq. (37). For $g_3 = g_1 + g_2$, we can get a solution by interchanging the states of the first and third qubit in Eq. (37). Choosing $g_1 = g_2 + g_3$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta = 1$, the dark-like state (37) corresponds to the horizontal line $E^- = 1$ in Figure 1.

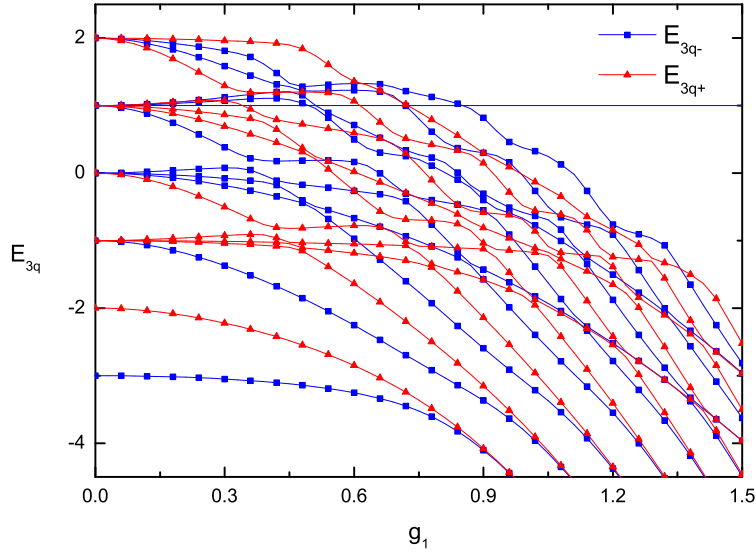


Figure 1. The numerical spectrum of three-qubit quantum Rabi model with $\Delta_1 = \Delta_2 = \Delta_3 = 1$, $\omega = 1$, $g_3 = 0.5g_2$, $0 \leq g_1 = g_2 + g_3 \leq 1.5$. E_+ and E_- are solutions with even and odd parity respectively.

Now we turn to the case of 4 qubit, with $(|\psi\rangle_{4q+}) = (|\psi\rangle_{3q-} \otimes |\downarrow\rangle_4, |\psi\rangle_{3q+} \otimes |\uparrow\rangle_4)$, $(|\psi\rangle_{4q-}) = (|\psi\rangle_{3q+} \otimes |\downarrow\rangle_4, |\psi\rangle_{3q-} \otimes |\uparrow\rangle_4)$, and

$$O_4 = \begin{pmatrix} g_1 & g_2 & g_3 & 0 & g_4 & 0 & 0 & 0 \\ g_2 & g_1 & 0 & g_3 & 0 & g_4 & 0 & 0 \\ g_3 & 0 & g_1 & g_2 & 0 & 0 & g_4 & 0 \\ 0 & g_3 & g_2 & g_1 & 0 & 0 & 0 & g_4 \\ g_4 & 0 & 0 & 0 & g_1 & g_2 & g_3 & 0 \\ 0 & g_4 & 0 & 0 & g_2 & g_1 & 0 & g_3 \\ 0 & 0 & g_4 & 0 & g_3 & 0 & g_1 & g_2 \\ 0 & 0 & 0 & g_4 & 0 & g_3 & g_2 & g_1 \end{pmatrix}, \quad (38)$$

As seen from the system of linear homogeneous equations (22), there are 24 rows and just 16 columns in its coefficient matrix. A solution exists if the nonzero rows is less than the columns in its row echelon form. First we consider the 8×8 diagonal matrix in (22)

$$D_{40}^{\pm} - E^{\pm} = \begin{pmatrix} D_{30}^{\mp} - \Delta_4 - E^{\pm} & 0 \\ 0 & D_{30}^{\pm} + \Delta_4 - E^{\pm} \end{pmatrix}, \quad (39)$$

and then take into account the other part

$$\begin{pmatrix} O_{40} & D_{41}^{\pm} - E^{\pm} \\ 0 & O_{41} \end{pmatrix} \begin{pmatrix} c_{4,0}^{\pm} \\ c_{4,1}^{\pm} \end{pmatrix} = 0. \quad (40)$$

We can simplify the condition $O_{41} c_{4,1}^{\pm} = 0$ by setting one of the eigenvalues of O_4 to be 0 and $(c_{4,1}^{\pm})$ to be its null vector (shown in table 1). This will eliminate 8 rows and 7 columns in the coefficient matrix in Eq. (40). If all the coupling strengths g_i ($i = 1, 2, 3, 4$) are nonzero, then there are at least 7 nonzero rows in the row echelon form of this coefficient matrix, because the number of the zero rows in the echelon form of O_{40} is just the same as its null vectors. Together with the diagonal matrix $D_{40}^{\pm} - E^{\pm}$, there are at least 15 rows but only 9 columns totally, so there should be at least 7 zero rows in $D_{40}^{\pm} - E^{\pm}$, which is impossible by analyzing Eq. (39).

It seems that there are no dark-like solutions for the 4-qubit Rabi model up to now. However, there are other possibilities by setting more than 1 eigenvalues in table 1 to be 0 simultaneously, which will eliminant more rows and less columns because there are more null vectors for O_4 . By analyzing table 1, we can choose $g_1 = g_2$ and $g_3 = g_4$ to set the eigenvalues $g_1 - g_2 + g_3 - g_4$ and $g_1 - g_2 - g_3 + g_4$ to be 0 simultaneously, then $(c_{4,1}^{\pm}|\psi_{4q\mp}\rangle)$ can be the linear superposition of the corresponding two null vectors (shown in Tab. 1), and there will be two variables. After elementary row transformation, the coefficient matrix in Eq. (40) reduces to row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1 \pm \Delta_1 - \Delta_2 - E^{\pm}}{g_1} + \frac{1 - \Delta_3 + \Delta_4 - E^{\pm}}{g_3} & \frac{1 \pm \Delta_1 - \Delta_2 - E^{\pm}}{g_1} + \frac{\Delta_3 - \Delta_4 + E^{\pm} - 1}{g_3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{-1 + \Delta_3 - \Delta_4 + E^{\pm}}{g_3} + \frac{E^{\pm} - 1}{g_1 - g_3} & \frac{1 - \Delta_3 + \Delta_4 - E^{\pm}}{g_3} + \frac{E^{\pm} - 1}{g_1 + g_3} \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & \frac{1 - \Delta_3 - \Delta_4 - E^{\pm}}{g_3} + \frac{1 \pm \Delta_1 + \Delta_2 - E^{\pm}}{g_1 - g_3} & \frac{1 - \Delta_3 - \Delta_4 - E^{\pm}}{g_3} - \frac{1 \pm \Delta_1 + \Delta_2 - E^{\pm}}{g_1 + g_3} \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \frac{-1 + \Delta_3 + \Delta_4 + E^{\pm}}{g_3} + \frac{E^{\pm} - 1}{g_3 - g_1} & \frac{-1 + \Delta_3 + \Delta_4 + E^{\pm}}{g_3} + \frac{E^{\pm} - 1}{g_1 + g_3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \frac{-1 \mp \Delta_1 - \Delta_2 + E^{\pm}}{g_1} + \frac{E^{\pm} - 1}{g_3 - g_1} & \frac{1 \pm \Delta_1 + \Delta_2 - E^{\pm}}{g_1} + \frac{E^{\pm} - 1}{g_1 + g_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \frac{-1 \mp \Delta_1 + \Delta_2 + E^{\pm}}{g_1} + \frac{E^{\pm} - 1}{g_1 - g_3} & \frac{-1 \mp \Delta_1 + \Delta_2 + E^{\pm}}{g_1} + \frac{E^{\pm} - 1}{g_1 + g_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4E^{\pm} - 4 & 4 - 4E^{\pm} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 - 4E^{\pm} & 4 - 4E^{\pm} \end{pmatrix}. \quad (41)$$

If $E^\pm = 1$, Eq. (41) reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{\pm\Delta_1-\Delta_2}{g_1} + \frac{-\Delta_3+\Delta_4}{g_3} & \frac{\pm\Delta_1-\Delta_2}{g_1} + \frac{\Delta_3-\Delta_4}{g_3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\Delta_3-\Delta_4}{g_3} & \frac{-\Delta_3+\Delta_4}{g_3} \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -\frac{\Delta_3+\Delta_4}{g_3} + \frac{\pm\Delta_1+\Delta_2}{g_1} & \frac{-\Delta_3-\Delta_4}{g_3} - \frac{\pm\Delta_1+\Delta_2}{g_1} \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \frac{\Delta_3+\Delta_4}{g_3} & \frac{\Delta_3+\Delta_4}{g_3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\frac{\pm\Delta_1+\Delta_2}{g_1} & \frac{\pm\Delta_1+\Delta_2}{g_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \frac{\mp\Delta_1+\Delta_2}{g_1} & \frac{\mp\Delta_1+\Delta_2}{g_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (42)$$

then together with $D_{40}^\pm - E^\pm$, there are 14 rows but just 10 columns totally, so there should be 5 zero rows in $D_{40}^\pm - E^\pm$, which seems impossible by only analyzing Eq. (39), but this is indeed not the case. For even parity, if $\Delta_1 - \Delta_2 = \pm 1 = \pm E$ and $\Delta_3 = \Delta_4$, there are dark-like state solutions by analyzing Eqs. (42) and (39),

$$|\psi\rangle_{g1} = \frac{1}{\mathcal{N}} \left(\frac{2(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad (43)$$

$$|\psi\rangle_{g2} = \frac{1}{\mathcal{N}} \left(\frac{2(\Delta_1 + \Delta_2)}{g} |0, \downarrow, \uparrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad (44)$$

where the first two qubits form a two-qubit dark-like state (29) and (30) respectively, and another two qubits form a spin singlet dark state. For odd parity, if $\Delta_1 + \Delta_2 = 1 = E^-$, there are similar dark-like states formed by $|\psi\rangle_e \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, where $|\psi\rangle_e$ is given by Eq. (27).

If $E^\pm \neq 1$, then Eq. (41) reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (45)$$

For even parity, if $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$, there is a “dark state” solution

$$|\psi\rangle_d = |0, \uparrow, \downarrow\rangle - |0, \downarrow, \uparrow\rangle \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad (46)$$

which is just the product of the two-qubit singlet.

Finally, we come to the case $g_1 = g_2 = g_3 = g_4 = g$. Now three eigenvalues $g_1 - g_2 + g_3 - g_4$, $g_1 - g_2 - g_3 + g_4$ and $g_1 + g_2 - g_3 - g_4$ are set to be 0, and there are three null vectors shown in table 1, which can be simplified to $(1, 0, 0, -1, -1, 0, 0, 1)^T$, $(0, 1, 0, -1, -1, 0, 1, 0)^T$, $(0, 0, 1, -1, -1, 1, 0, 0)^T$. Supposing that $(c_{4,1}^\pm |\psi_{4q\mp}\rangle)$ is the linear superposition of these null vectors, after elementary row transformation, the coefficient matrix

in Eq. (40) reduces to row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{3\pm\Delta_1-\Delta_4-3E^\pm}{g} & \frac{2+\Delta_2-\Delta_4-2E^\pm}{g} & \frac{5+\Delta_3-\Delta_4-5E^\pm}{g} \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{2-\Delta_2-\Delta_3-2E^\pm}{g} & \frac{1\mp\Delta_1-\Delta_3-E^\pm}{g} & \frac{3-\Delta_3+\Delta_4-3E^\pm}{g} \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & \frac{2-\Delta_2-\Delta_3-2E^\pm}{g} & \frac{2-\Delta_2+\Delta_4-2E^\pm}{g} & \frac{2\mp\Delta_1-\Delta_2-2E^\pm}{g} \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & \frac{-3\mp\Delta_1+\Delta_2+\Delta_3+\Delta_4+3E^\pm}{g} & \frac{2E^\pm-2}{g} & \frac{2E^\pm-2}{g} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \frac{-2+\Delta_2+\Delta_3+2E^\pm}{g} & \frac{-1\pm\Delta_1+\Delta_3+E^\pm}{g} & \frac{-2\pm\Delta_1+\Delta_2+2E^\pm}{g} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2-2E^\pm}{g} & \frac{4-4E^\pm}{g} & \frac{2-2E^\pm}{g} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2E^\pm-2}{g} & \frac{2-2E^\pm}{g} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8E^\pm-8}{g} \end{pmatrix}. \quad (47)$$

If $E^\pm = 1$, Eq. (47) reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{\pm\Delta_1-\Delta_4}{g} & \frac{\Delta_2-\Delta_4}{g} & \frac{\Delta_3-\Delta_4}{g} \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{-\Delta_2-\Delta_3}{g} & \frac{\mp\Delta_1-\Delta_3}{g} & \frac{-\Delta_3+\Delta_4}{g} \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & \frac{-\Delta_2-\Delta_3}{g} & \frac{-\Delta_2+\Delta_4}{g} & \frac{\mp\Delta_1-\Delta_2}{g} \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & \frac{\mp\Delta_1+\Delta_2+\Delta_3+\Delta_4}{g} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \frac{\Delta_2+\Delta_3}{g} & \frac{\pm\Delta_1+\Delta_3}{g} & \frac{\pm\Delta_1+\Delta_2}{g} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (48)$$

then together with $D_{40}^\pm - E^\pm$, there are 13 rows but just 11 columns totally, so there should be 3 zero rows in $D_{40}^\pm - E^\pm$, which is possible by choosing

$$\Delta_2 = \Delta_3 = \Delta_4 = \pm\Delta_1 - 1 \quad \text{or} \quad (49)$$

$$\Delta_2 = \Delta_3 = \Delta_4 = \pm\Delta_1 + 1. \quad (50)$$

We can interchange Δ_2 with Δ_1 , Δ_3 and Δ_4 , so there are totally 8 choices, but we only consider (49) and (50), because they are equivalent.

Substituting Eq. (49) into Eq. (48), for even parity, we obtain one dark-like eigenstate

$$\begin{aligned} |\psi\rangle_{g1} = & a \left(\frac{(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right)_{1,2} \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)_{3,4} \\ & + b \left(\frac{(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right)_{1,3} \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)_{2,4} \\ & + c \left(\frac{(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right)_{1,4} \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)_{2,3}. \end{aligned} \quad (51)$$

This can be easily understood due to the fact that for $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_1 - 1$, there are three independent solutions, each formed by the product of a two-qubit dark-like state and a two-qubit singlet. For $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_1 + 1$, the solution takes the same form as (51) with the dark-like state substituted by (30). For even parity, we choose $\Delta_2 = \Delta_3 = \Delta_4 = -\Delta_1 + 1$, and the dark-like state takes the same form as (51) with the dark-like state substituted by (27). Choosing $\Delta_1 = \Delta_2 + 1$, $\Delta_3 = \Delta_4$, $g_1 = g_2$, $g_3 = g_4$, a dark-like state (43) corresponding to the horizontal line $E^+ = 1$ is shown in Figure 2.

We can follow the similar procedure to find dark-like states for 5, 6, 7, ... qubits cases. The key point is just to solve (22) with different format. It should be pointed out that we haven't found a universal existence condition and all the dark-like states for arbitrary qubit number N, because it still needs detailed analysis for more qubits. But we find one kind of

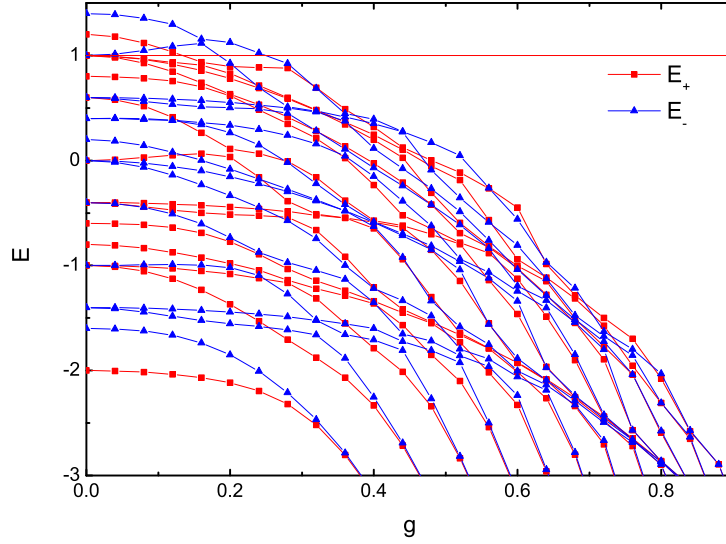


Figure 2. The numerical spectrum of the four-qubit quantum Rabi model with $\Delta_1 = \Delta_2 + 1 = 1.2$, $\Delta_3 = \Delta_4 = 0.3$, $\omega = 1$, $g_1 = g_2 = g_3 = g_4$, $0 \leq g_1 = g_2 = g_3 = g_4 = g \leq 1$. E_+ and E_- are solutions with even and odd parity respectively.

dark-like states commonly exist for arbitrary qubit number $N > 1$

$$|\psi\rangle_{Nqdark-like} = |\psi\rangle_{2qdark-like} \times (|\psi\rangle_{singlet})^{(N-2)/2} \quad N = 2, 4, 6, 8, \dots \quad (52)$$

$$|\psi\rangle_{Nqdark-like} = |\psi\rangle_{3qdark-like} \times (|\psi\rangle_{singlet})^{(N-3)/2} \quad N = 3, 5, 7, 9, \dots \quad (53)$$

$N = 4$ is an example of Eq. (52), and all its eigenstates have the form of Eq. (52).

3. Dark-like states for the multi-qubit and multi-photon Rabi model

The N -qubit and M -photon Rabi model reads

$$H_{PQ} = \omega a^\dagger a + \sum_{i=1}^N g_i \sigma_{ix} (a^M + a^{\dagger M}) + \sum_{i=1}^N \Delta_i \sigma_{iz}, \quad (54)$$

where M is a positive integer. This model is of considerable interest because of its relevance to the study of the coupling between multi-qubit and photon field with the qubit making M -photon transitions. Besides, it is known that under rotating wave approximation, the dynamics of the M -photon J-C model [34] is qualitatively different from that of the usual single-photon case [33]. As discussed in Ref. [33, 35, 36], for single qubit case, this model is solvable only if $M \leq 2$ and the coupling parameter is below a certain critical value. But in the following discussion, we will show that the case for more qubits is different: Dark-like eigenstates for H_{PQ} (54) with $N > 1$ still exist, regardless of these constraints, although in usual cases this model is indeed not well-defined.

However, we first try to find out this critical value for $M = 2$. We assume that $\Delta_k (k = 1, 2, \dots, N) = 0$, which does not affect the result [35]. In the basis formed by

the eigenstates of $\prod \sigma_{jx} (j = 1, 2, \dots, N)$, the Hamiltonian (54) with $M = 2$ is turned into the form [35]

$$h_{PQ} = a^\dagger a + \lambda(a^M + a^{\dagger M}), \quad (55)$$

where $\lambda = \pm g_1 \pm g_2 \dots \pm g_N$. Defining operators

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad p = i\sqrt{\frac{1}{2}}(a^\dagger - a), \quad (56)$$

then h_{PQ} can be rewritten as

$$h_{PQ} = p^2 + \frac{1 + 2\lambda}{1 - 2\lambda}x^2 - \frac{1}{2}. \quad (57)$$

Clearly, if $\frac{1+2\lambda}{1-2\lambda} = \omega^2 > 0$, then h_{PQ} corresponds to a quantum harmonic oscillator and can be diagonalized. However, if $\frac{1+2\lambda}{1-2\lambda} = -\omega^2 < 0$, h_{PQ} represents an inverted quantum harmonic oscillator, which cannot be diagonalized using the basis states $|n\rangle$ of the number operator because its eigenstates are not normalizable. Thus the condition for the Hamiltonian (55) being diagonalizable is $\lambda < \frac{1}{2}$ [35], and correspondingly we have $\max\{\pm g_1 \pm g_2 \dots \pm g_N\} < \frac{1}{2}$, that is

$$\sum_{k=1}^N g_k < \frac{1}{2}. \quad (58)$$

Then we search for the dark-like eigenstates for H_{PQ} (54). There are $2M$ invariant subspaces

$$\begin{aligned} & \{|0, \psi_{Nq+}\rangle, |M, \psi_{Nq-}\rangle, |2M, \psi_{Nq+}\rangle, \dots\} \\ & \{|0, \psi_{Nq-}\rangle, |M, \psi_{Nq+}\rangle, |2M, \psi_{Nq-}\rangle, \dots\} \\ & \{|1, \psi_{Nq+}\rangle, |M+1, \psi_{Nq-}\rangle, |2M+1, \psi_{Nq+}\rangle, \dots\} \\ & \{|1, \psi_{Nq-}\rangle, |M+1, \psi_{Nq+}\rangle, |2M+1, \psi_{Nq-}\rangle, \dots\} \\ & \dots \\ & \{|M-1, \psi_{Nq+}\rangle, |2M-1, \psi_{Nq-}\rangle, |3M-1, \psi_{Nq+}\rangle, \dots\} \\ & \{|M-1, \psi_{Nq-}\rangle, |2M-1, \psi_{Nq+}\rangle, |3M-1, \psi_{Nq-}\rangle, \dots\}, \end{aligned} \quad (59)$$

each of which can be labeled by $\{i, \pm\}$, where the initial photon number takes the values $i = 0, 1, 2, \dots, M-1$, and \pm is the eigenvalue of $\prod_{k=1}^N \sigma_{kz}$ for the initial qubit state. H_{PQ} in each subspace has the same form as H_{NQ} (22) except for some constants

$$H_{PQi}^\pm = \begin{pmatrix} i + D_{n0}^\pm & \sqrt{\frac{(i+M)!}{i!}} O_{n0}^\pm & 0 & 0 & 0 & \dots \\ \sqrt{\frac{(i+M)!}{i!}} O_{n0}^\pm & i + M - 1 + D_{n1}^\pm & \sqrt{\frac{(i+2M)!}{2(i+M)!}} O_{n1}^\pm & 0 & 0 & \dots \\ 0 & \sqrt{\frac{(i+2M)!}{2(i+M)!}} O_{n1}^\pm & i + 2M - 2 + D_{n2}^\pm & \sqrt{\frac{(i+3M)!}{3(i+2M)!}} O_{n2}^\pm & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (60)$$

where D_{nj}^\pm and O_{nj}^\pm ($j = 0, 1, 2, 3, \dots$) are just the same as defined in the N-qubit Rabi model in (8) and (9), respectively.

Now, to find out the dark-like solution, we follow the steps for the N-qubit case to get

$$\begin{pmatrix} i + D_{n0}^\pm - E_i^\pm & 0 \\ \sqrt{\frac{(i+M)!}{i!}} O_{n0}^\pm & i + M - 1 + D_{n1}^\pm - E_i^\pm \\ 0 & \sqrt{\frac{(i+M)!}{i!}} O_{n1}^\pm \end{pmatrix} \begin{pmatrix} c_{n,0}^\pm \\ c_{n,1}^\pm \end{pmatrix} = 0. \quad (61)$$

If we define $E^\pm = E_i^\pm - (i + M - 1)$ and $g_k = \sqrt{\frac{i!}{(i+M)!}} g_{i,k}$ ($k = 1, 2, \dots, N$), so that $\sqrt{\frac{(i+M)!}{i!}} O_{n,0,1}^\pm \rightarrow O_{n,0,1}^\pm$, then we obtain

$$\begin{pmatrix} D_{n0}^\pm - (E^\pm + M - 1) & 0 \\ O_{n0}^\pm & D_{n1}^\pm - E^\pm \\ 0 & O_{n1}^\pm \end{pmatrix} \begin{pmatrix} c_{n,0}^\pm \\ c_{n,1}^\pm \end{pmatrix} = 0. \quad (62)$$

Eq. (62) has exactly the same form as Eq. (22), except for $[D_{n0}^\pm - (E^\pm + M - 1)]c_{n,0}^\pm = 0$, which will just determine the relation between Δ_k ($k = 1, 2, \dots, N$) and E^\pm , so we can get the dark-like solution for Eq. (62) from the solution to Eq. (22) for the N-qubit Rabi model just by making the replacement $f(\Delta_k, E^\pm) = 0 \rightarrow f(\Delta_k, (E^\pm + M - 1)) = 0$. To conclude, for a dark-like state of the N-qubit Rabi model, we can get a corresponding dark-like state of the N-qubit and M-photon Rabi model in the subspace labeled by $\{i, \pm\}$, upon using the following relations

$$E_i^\pm = E^\pm + i + M - 1 \quad (63)$$

$$g_{i,k} = \sqrt{\frac{(i+M)!}{i!}} g_k (k = 1, 2, \dots, N) \quad (64)$$

$$f(\Delta_{i\pm,k}, E_i^\pm - i) = f(\Delta_k, E^\pm) = 0. \quad (65)$$

As discussed above, the dark-like eigenstates of H_{PQ} (54) exist for arbitrary photon number M in the whole qubit-photon coupling regime with constant energy, even though generally the model is only solvable under some constraints on the coupling strength and photon number M .

For the two-qubit and two-photon Rabi model, there are six dark-like states

$$|\psi\rangle_{0,+} = \frac{2(\Delta_1 - \Delta_2)}{\sqrt{2}g} |0, \uparrow, \uparrow\rangle - |2, \uparrow, \downarrow\rangle + |2, \downarrow, \uparrow\rangle, \quad (66)$$

$$|\psi\rangle_{1,+} = \frac{2(\Delta_1 - \Delta_2)}{\sqrt{6}g} |1, \uparrow, \uparrow\rangle - |3, \uparrow, \downarrow\rangle + |3, \downarrow, \uparrow\rangle, \quad (67)$$

with the conditions $g_1 = g_2 = g/2$, $\Delta_1 + \Delta_2 = 2$ and $E^+ = 2, 3$ respectively, and

$$|\psi\rangle_{0,-,a} = \left(\frac{2(\Delta_1 + \Delta_2)}{\sqrt{2}g} |0, \uparrow, \downarrow\rangle + |2, \downarrow, \downarrow\rangle - |2, \uparrow, \uparrow\rangle \right), \quad (68)$$

$$|\psi\rangle_{1,-,a} = \left(\frac{2(\Delta_1 + \Delta_2)}{\sqrt{6}g} |1, \uparrow, \downarrow\rangle + |3, \downarrow, \downarrow\rangle - |3, \uparrow, \uparrow\rangle \right), \quad (69)$$

with the conditions $g_1 = g_2 = g/2$, $\Delta_1 - \Delta_2 = 2$ and $E^- = 2, 3$ respectively, and

$$|\psi\rangle_{0,-,b} = \left(\frac{2(\Delta_1 + \Delta_2)}{\sqrt{2}g} |0, \downarrow, \uparrow\rangle + |2, \downarrow, \downarrow\rangle - |2, \uparrow, \uparrow\rangle \right), \quad (70)$$

$$|\psi\rangle_{1,-,b} = \left(\frac{2(\Delta_1 + \Delta_2)}{\sqrt{6}g} |1, \downarrow, \uparrow\rangle + |3, \downarrow, \downarrow\rangle - |3, \uparrow, \uparrow\rangle \right), \quad (71)$$

with the conditions $g_1 = g_2 = g/2$, $\Delta_2 - \Delta_1 = 2$ and $E^- = 2, 3$ respectively.

Choosing $g_1 = g_2 = g/2$, $\Delta_1 + \Delta_2 = 2$, the spectrum of the two-qubit and two-photon Rabi model is shown in Figure 3, where the dark-like states (66) and (67) correspond to

the horizontal line $E^+ = 2$ and $E^+ = 3$ respectively. These special states exist in the whole coupling regime, while other eigenstates exist only for $g < 0.5$. Besides, they commonly exist even for multi-qubit and M-photon ($M > 2$) Rabi model. This can be tested by numerical diagonalization: Although the eigenvalues usually will not converge for $M = 3$, with regard to dark-like states, the eigenvalue always converge at $E = 3$ with $i = 0$.

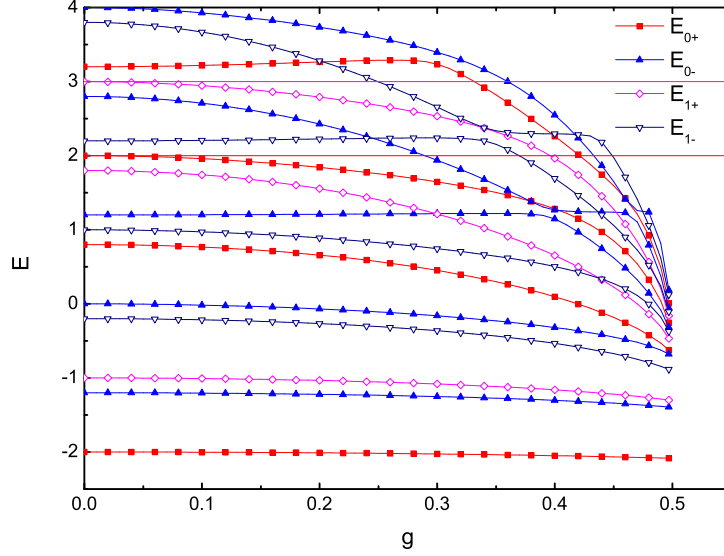


Figure 3. The numerical spectrum of two-qubit quantum Rabi model with $\Delta_1 = 1.6$, $\Delta_2 = 0.4$, $\omega = 1$, $g_1 = g_2 = g/2$, $0 \leq g \leq 0.5$. E_{0+} , E_{0-} , E_{1+} , E_{1-} are eigenvalues of four invariant subspace labeled by (i, \pm) respectively.

4. Experimental considerations

In the past few years there have appeared a series of proposals for the implementation of the quantum Rabi model in all its parameter regimes, via analog or digital-analog quantum simulations, in a variety of quantum platforms including trapped ions [37, 38] and superconducting circuits [39]. Moreover, the multiqubit, single-photon Rabi model may be straightforwardly implemented in superconducting circuits via a digital-analog quantum simulator [40, 41]. Indeed, a set of superconducting qubits capacitively coupled with a coplanar microwave resonator naturally implement a Tavis-Cummings Hamiltonian. Via digital-analog techniques, one can combine this naturally-appearing interaction with local rotations, in order to reproduce the multiqubit Rabi model in all parameter regimes, and with arbitrary inhomogeneous couplings and qubit energies, with polynomial resources [40, 41]. Therefore, a quantum dynamics provided by the Hamiltonian in Eq. (1) can be carried out in the lab with current technology. In order to probe the dark-like states of the multiqubit, single-photon Rabi model, one may proceed initializing the system in an eigenstate of an easy

to initialize Hamiltonian, e.g., the purely qubit and bosonic mode free terms without mutual interaction, and adiabatically turn on the multiqubit Rabi coupling term, via a digitization of the adiabatic evolution, as in Ref. [42]. In order to measure the energy, to check its constant character under parameter change, one may either apply the phase estimation algorithm, or measure term by term of the Hamiltonian, with standard superconducting circuit technology [40, 41].

5. Conclusions

We have found dark-like states for multi-qubit and multi-photon Rabi models, which exist in the whole coupling regime with constant eigenenergy, with qubit and photon field still being coupled. Besides, their photon numbers are bounded from above, distinctly different from the one qubit case, because there are closed subspaces in Fock space due to the interaction between multi-qubit and photon field. Their existence conditions are simple, which does not depend on qubit energy and coupling strength at the same time. And they correspond to horizontal lines in the spectra, which means for arbitrary coupling g_i , we always find one such state by tuning other conditions. These dark-like states can also serve as benchmarks for numerical techniques and as foundations for perturbative treatments.

For the single-qubit and multi-photon Rabi model, the solution exists only if the photon number $M \leq 2$ and the coupling strength is below a certain critical value. But multi-qubits make it different. There exist dark-like eigenstates in the whole coupling regime for arbitrary M under certain conditions. This is due to the closed subspace in the photon number representation brought about by the multi-qubit, so just like the multi-photon J-C model, the multi-photon Rabi model is diagonalizable in this special case.

Dark states can preserve entanglement under dissipation, driving and dipole-dipole interactions, so they can be used to store correlations. Dark-like states have similar properties as dark states in the spectra, but their properties under the influence of environment (dissipation, dephasing, or the like) need to be explored. Whether this kind of dark-like states has similar applications as dark-like states or has other peculiarities is a very interesting problem to study.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grants Nos 11535004, 11347112, 11204263, 11035001, 11404274, 10735010, 10975072, 11375086 and 11120101005), by the 973 National Major State Basic Research and Development of China (Grants Nos 2010CB327803 and 2013CB834400), by CAS Knowledge Innovation Project No. KJCX2-SW-N02, by Research Fund of Doctoral Point (RFDP) Grant No. 20100091110028, by the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), by the Scientific Research Fund of Hunan Provincial Education Department (No. 12C0416), by the Program for Changjiang Scholars

and Innovative Research Team in University (IRT13093), by Spanish MINECO/FEDER FIS2015-69983-P, UPV/EHU UFI 11/55, and Ramón y Cajal Grant RYC- 2012-11391.

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